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# ON SOME GEOMETRICAL PROPERTIES OF SEIFERT BUNDLES

BY

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ABSTRACT

In this paper we use the integration along the leaves introduced by Haefliger in 1980 to obtain a differentiable version of the Gysin sequence and Euler class for compact Hausdorff orientable foliations with generic leaf the sphere $S^p$. From this we give a geometrical significance to the vanishing of the Euler class on Seifert bundles. We also obtain an integral formula on Seifert bundles similar to the Gauss–Bonnet one.

## §1. Introduction

In [3] Haefliger defines for each oriented foliation $\mathcal{F}$, a linear operator $\int_{\mathcal{F}}$ (the integration along the leaves) which has similar properties to those of the integration over the fibres on fibre bundles.

If $\mathcal{F}$ is Hausdorff and compact this operator can be interpreted as a linear map of degree $-p$

$$\int_{\mathcal{F}} : A^*(M) \to A^*(M/\mathcal{F})$$

where $A(M/\mathcal{F})$ denotes the algebra of $V$-forms on the $V$-manifold $M/\mathcal{F}$.

In this paper we use the integration along the leaves to obtain a differentiable version of the Gysin exact sequence and Euler class for compact Hausdorff orientable foliations with generic leaf the sphere $S^p$.

From this interpretation we obtain the main theorems of this paper.

THEOREM A. *Let $M$ be a compact oriented manifold and $\mathcal{F}$ an oriented Hausdorff foliation by circles on $M$. The following conditions are equivalent*:

(i) *The Euler class of $\mathcal{F}$ is zero.*

(ii) *There exists a Riemannian foliation complementary to $\mathcal{F}$* (see Definition 2).

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THEOREM B. *Let $M$ be a compact orientable 3-manifold and let $\mathcal{F}$ be an orientable Seifert bundle on $M$. The following conditions are equivalent*:

(i) *The first Betti number of $M$ is even.*

(ii) *The Euler class of $\mathcal{F}$ is not zero.*

(iii) *$\mathcal{F}$ is a contact foliation* (see Definition 3).

(iv) *$\mathcal{F}$ does not admit any complementary Riemannian foliation.*

We also show that the implication (ii) $\Rightarrow$ (iv) is true for an oriented $p$-dimensional sphere foliation on an $n$-dimensional manifold $M$ (see Proposition 3). We give an example of an oriented $p$-dimensional sphere foliation $\mathcal{F}$ with complementary Riemannian foliation $\mathcal{F}^{\perp}$ such that neither $\mathcal{F}$ nor $\mathcal{F}^{\perp}$ are fibrations.

In the last paragraph, and using also the integration along the leaves, we obtain an integral formula on Seifert bundles similar to the Gauss–Bonnet one.

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We are also grateful to the referee for suggesting the current proof of Proposition 5 which, in the original version of this paper, was proved by using a metric.

## §2. Integration along the leaves and Euler class

2.1. In this paper $M$ will denote an oriented compact connected smooth manifold of dimension $p+q$.

Let $\mathcal{F}$ be a codimension $q$ smooth oriented Hausdorff compact foliation on $M$, i.e., all leaves of $\mathcal{F}$ are compact, smoothly oriented, and the leaf space $B = M/\mathcal{F}$ is Hausdorff.

It is well known (cf. for instance [2]) that $\mathcal{F}$ admits the following local representation:

THEOREM 1. *There is a generic leaf $L$ and an open dense subset of $M$ where the leaves are all diffeomorphic to $L$. Moreover, given a leaf $L_0$, there is*

(a) *a finite subgroup $G$ of $O(q)$,*

(b) *a free action of $G$ on $L$, on the right,*

(c) *an open neighborhood $V$ of $L_0$,*

(d) *a diffeomorphism $\phi : L \times_G D \to V$, where $D$ is the unit ball of $\mathbf{R}^q$, which preserves leaves if one takes the foliation on $L \times_G D$ whose leaves are the quotient of the submanifolds $L \times \{point\}$.*

It follows from this theorem that $B$ is, in a natural way, a $V$-manifold and that the canonical projection $\pi : M \to B$ defines a $V$-fibre space structure over $B$. For definitions we refer to [8].

In [3] Haefliger defines, for each oriented $p$-dimensional foliation $\mathcal{F}$ on $M$, the integration along the leaves as a linear map

$$\int_{\mathcal{F}} : A_c^{p+k}(M) \to A_c^k(\mathrm{Tr}\,\mathcal{F})$$

where $A_c^k(\mathrm{Tr}\,\mathcal{F})$ denotes the quotient of the vector space of $k$-forms with compact support on a submanifold $T$ transversal to every leaf of $\mathcal{F}$ by the vector subspace generated by elements of the form $\alpha - h^*\alpha$, where $h$ belongs to the holonomy pseudogroup of $\mathcal{F}$ and $\alpha$ is a $k$-form on $T$.

In our situation (oriented Hausdorff compact foliations) this operator can be interpreted as a linear map of degree $-p$, $\int_{\mathcal{F}} : A^*(M) \to A^*(B)$, where $A^*(B)$ denotes the algebra of $V$-forms on $B$. Moreover, if $\omega$ is a $k$-form on $M$, we can compute $\int_{\mathcal{F}} \omega$ as follows: In each local model $L \times_G D$ the $k$-form $\phi^*\omega$ can be regarded as a $G$-invariant $k$-form on $L \times D$. The integration of this form along the fibres in the trivial bundle $L \times D \to D$ gives us a $G$-invariant $(k-p)$-form on $D$ which we divide by the order of $G$ to get a consistent construction. Now the canonical structure of $V$-fibre space of $\pi : M \to B$ tells us that the above construction defines a $V$-form on the $V$-manifold $B$. This $V$-form is $\int_{\mathcal{F}} \omega$.

Note that if $\mathcal{F}$ is a fibration this definition coincides with the usual one.

We list here some properties of this operator we shall use later. All the statements follow directly from the analogous properties of the integration over the fibres on fibre bundles.

PROPOSITION 1. (a) *The integration along the leaves commutes with the exterior derivative.*

(b) *Let $\tau$ be a $r-V$-form on $B$ and $\omega$ an $s$-form on $M$. Then $\int_{\mathcal{F}}(\pi^*\tau \wedge \omega) = \tau \wedge \int_{\mathcal{F}} \omega$.*

(c) *With respect to the orientation of $B$ induced by the orientations of $M$ and the leaves we have $\int_M \omega = \int_B (\int_{\mathcal{F}} \omega)$ where $\omega$ is a $n$-form on $M$.*

2.2. We recall here the definition of a bundle-like metric [5]. Let $g$ be a Riemannian metric on $M$. For each flat neighborhood $(U, x^1, \cdots, x^p, y^1, \cdots, y^q)$ of a foliation $\mathcal{F}$ (the foliation in $U$ is given by $y^\alpha = \text{constant}$), there are 1-forms $\theta^1, \cdots, \theta^p$ such that $\{\theta^1, \cdots, \theta^p, dy^1, \cdots, dy^q\}$ is a basis of the cotangent space with dual basis $(\partial/\partial x^1, \cdots, \partial/\partial x^p, v_1, \cdots, v_q\}$, where $v_\alpha$ are vector fields orthogonal to $\mathcal{F}$ by $g$. We say $g$ is bundle-like with respect to $\mathcal{F}$ if in each flat neighborhood we have

$$g = g_{ij}(x, y)\theta^i\theta^j + g_{\alpha\beta}(y)dy^\alpha dy^\beta.$$

If $\mathcal{F}$ is a compact foliation and there exists a bundle-like metric on $M$ with respect to $\mathcal{F}$, then $\mathcal{F}$ is Hausdorff [5]. Reciprocally, if $\mathcal{F}$ is Hausdorff and compact, one can obtain a bundle-like metric by glueing together, by means of a partition of unity on $M$ constant on the leaves, bundle-like metrics on the local models.

Note that a bundle-like metric induces a Riemannian metric invariant by the holonomy pseudogroup on each submanifold transverse to $\mathcal{F}$, and thus a Riemannian metric on the $V$-manifold $B$.

Let $g$ be a bundle-like metric on $M$ with respect to a foliation $\mathcal{F}$. If on each flat neighborhood the 1-forms $\theta^1, \cdots, \theta^p$ are positively ordered, then

$$\eta = \sqrt{\det(g_{ij})}\,\theta^1 \wedge \cdots \wedge \theta^p$$

is a global $p$-form on $M$. The restriction of $\eta$ to each leaf $L_0$ of $\mathcal{F}$ gives the volume element of $L_0$ with respect to the metric induced on $L_0$ by $g$. $\eta$ is called the volume form associated to $g$ and to the orientation of $\mathcal{F}$.

Definition 1. A bundle-like metric will be called of constant volume if all regular leaves (leaves with trivial holonomy group) have the same volume.

For such metrics the leaves are minimal (cf. Rummler, Comm. Math. Helv., **54**, 224–239).

With the notation above, $g$ is of constant volume if and only if the function $\int_{\mathcal{F}} \eta$ is constant.

We obtain such a metric from a bundle-like metric $g'$ in the following way: Let $\eta'$ be the volume form associated to $g'$. Then $h' = \int_{\mathcal{F}} \eta'$ is a strictly positive function on $B$ and the desired metric is $g = h \,.\, g'$ where $h = (h' \circ \pi)^{-2/p}$. (Here $\int_{\mathcal{F}} \eta = 1$, where $\eta$ is the volume form associated to $g$.)

2.3. In this paragraph $\mathcal{F}$ will be an oriented sphere foliation, i.e., an oriented Hausdorff compact foliation with generic leaf the sphere $S$ of dimension $p$. We also assume $M$ compact.

The situation being similar to that of sphere bundles, we can give a differentiable version of the Euler class and Gysin exact sequence, via the integration along the leaves.

It follows from paragraph 2.2 that there exists $\eta \in A^p(M)$ such that $\int_{\mathcal{F}} \eta = 1$. Thus, for each $\tau \in A^*(B)$, we have $\int_{\mathcal{F}} (\pi^*\tau \wedge \eta) = \tau$, and $\int_{\mathcal{F}} : A^r(M) \to A^{r-p}(B)$ is surjective. Let $K^r$ denote the kernel of $\int_{\mathcal{F}}$. It follows from Proposition 1(a) that $d(K^r) \subset K^{r-1}$. Thus we obtain the exact sequence of differential spaces

$$0 \longrightarrow K^* \longrightarrow A^*(M) \xrightarrow{\int_{\mathcal{F}}} A^*(B) \to 0.$$

We denote by $\delta$ the connecting homomorphism associated to this sequence.

On the other hand, $\operatorname{Im} \pi^* \subset K^*$ and the commutative diagram

$$\begin{array}{ccc} A^r(B) & \xrightarrow{\pi^*} & K^r \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ A^{r+1}(B) & \xrightarrow{\pi^*} & K^{r+1} \end{array}$$

induces, for each $r$, a morphism $\pi_r^\# : H^r(B) \to H^r(K^*)$. (For the construction of the de Rham cohomology of a $V$-manifold see [7].)

Now an argument similar to the case of sphere bundles shows that $\pi^\#$ is an isomorphism (see, for instance, Greub, Halperin and Vanstone, *Connections, Curvature, and Cohomology*, Vol. I). From this and the above exact sequence we get the following proposition.

PROPOSITION 2. *If $\mathcal{F}$ is an oriented sphere foliation then the sequence (the Gysin sequence of $\mathcal{F}$)*

$$\cdots \longrightarrow H^r(B) \xrightarrow{\mathcal{D}} H^{r+p+1}(B) \xrightarrow{\pi^\#} H^{r+p+1}(M) \xrightarrow{\int_{\mathcal{F}}^\#} H^{r+1}(B) \longrightarrow \cdots$$

*where $\mathcal{D} = (\pi^\#)^{-1} \circ \delta$ is exact.*

Since $B$ is connected $H^0(B) \cong \mathbf{R}$ and we can define, as is usual, the Euler class of the sphere foliation $\mathcal{F}$ by $\chi_{\mathcal{F}} = \mathcal{D}(1) \in H^{p+1}(B)$.

REMARKS. (1) The connecting homomorphism $\mathcal{D}$ is nothing but the product by the Euler class $\chi_{\mathcal{F}}$. Moreover, if $\omega \in A^p(M)$ and $\tau \in A^{p+1}(B)$ are such that $\int_{\mathcal{F}} \omega = 1$ and $d\omega = \pi^* \tau$, then $\chi_{\mathcal{F}} = [\tau]$.

(2) As in the case of sphere bundles, we have that if the Euler class is zero then $H^*(M) \cong H^*(B) \otimes H^*(S^p)$.

We end this paragraph by computing the Euler class in a particular case.

DEFINITION 2. A codimension $p$ foliation $\mathcal{F}^\perp$ transverse to $\mathcal{F}$ is said to be a Riemannian foliation complementary to $\mathcal{F}$ if there is a Riemannian metric on $M$ bundle-like both with respect to $\mathcal{F}$ and $\mathcal{F}^\perp$.

Note that if $\mathcal{F}$ and $\mathcal{F}^\perp$ are Hausdorff compact, then $\mathcal{F}^\perp$ is a Riemannian foliation complementary to $\mathcal{F}$.

PROPOSITION 3. *Assume the sphere foliation $\mathcal{F}$ admits a Riemannian foliation $\mathcal{F}^\perp$ complementary to $\mathcal{F}$. Then the Euler class of $\mathcal{F}$ is zero.*

PROOF. Let $g$ be a Riemannian metric on $M$ bundle-like with respect to the two foliations $\mathcal{F}$ and $\mathcal{F}^\perp$. Let $\eta$ be the volume $p$-form associated to $g$. We can cover $M$ by local charts $(U, x^i, y^\alpha)$ such that the foliations $\mathcal{F}$ and $\mathcal{F}^\perp$ are given locally by $x^i$ = constant and $y^\alpha$ = constant, respectively. In such a chart we have

$$g = g_{ij}(x)dx^i dx^j + g_{\alpha\beta}(y)dy^\alpha dy^\beta$$

and

$$\eta = \sqrt{\det(g_{ij})}\,dx^1 \wedge \cdots \wedge dx^p$$

where $x^1, \cdots, x^p$ are ordered positively with respect to the orientation of $\mathcal{F}$. We have $d\eta = 0$, $h = \int_{\mathcal{F}} \eta > 0$ and $dh = \int_{\mathcal{F}} d\eta = 0$, so $h$ is constant, different from zero, and the morphism $\int_{\mathcal{F}}^{*}: H^p(M) \to H^0(B)$ is surjective. Thus the Euler class is zero.

COROLLARY 1. *If $M$ is a homology $n$-sphere, then any Seifert sphere foliation $\mathcal{F}$ does not admit any Riemannian foliation complementary to $\mathcal{F}$.*

PROOF. Use Proposition 2 and Remark 2.

REMARK 3. A Riemannian foliation complementary to $\mathcal{F}$ is not necessarily a fibration. Simple examples of this can by obtained from the action of a finite group on a product of manifolds. Take, for instance, $\tilde{M} = S^{2m-1} \times T^3$ $(S^{2m-1} = \{z = (z_1, \cdots, z_m) \in \mathbf{C}^m;\ \sum_1^m z_i \bar{z}_i = 1\},\ T^3 = \{(w_1, w_2, w_3) \in \mathbf{C}^3;\ w_i \bar{w}_i = 1,\ i = 1, 2, 3\})$. Consider now the action of $G = \mathbf{Z}_2 \oplus \mathbf{Z}_2 = \{1, a, b, c\}$ on $\tilde{M}$ given by

$$a(z, w_1, w_2, w_3) = (\bar{z}, -w_1, w_2, w_3),$$

$$b(z, w_1, w_2, w_3) = (-z, w_1, \bar{w}_2, \bar{w}_3),$$

$$c(z, w_1, w_2, w_3) = (-\bar{z}, -w_1, \bar{w}_2, \bar{w}_3),$$

where $\bar{z} = (\bar{z}_1, \cdots, \bar{z}_m)$.

This action is free and $M = \tilde{M}/G$ is a manifold. The foliations $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^\perp$ on $\tilde{M}$ whose leaves are respectively $S^{2m-1} \times \{\text{point}\}$ and $\{\text{point}\} \times T^3$ are preserved by the action of $G$. So we have foliations $\mathcal{F}$ and $\mathcal{F}^\perp$ on $M$. They are transverse Hausdorff compact foliations and they are not fibrations.

Note that if $m$ is even then $M$ and $\mathcal{F}$ are orientable.

## §3. Seifert bundles and Euler class

3.1. In this paragraph $\mathcal{F}$ will be a circle Hausdorff foliation, so the leaves of $\mathcal{F}$ are the fibres of a Seifert bundle. Our object here is to give some geometrical significance to the Euler class.

Recall that if $\mathcal{F}$ is an oriented circle foliation the following conditions are equivalent:

(i) $\mathcal{F}$ is Hausdorff.

(ii) There is an action $\rho : S^1 \times M \to M$ such that the leaves of $\mathcal{F}$ are the orbits of $\rho$.

(iii) There is some Riemannian metric with respect to which the orbits are totally geodesic submanifolds.

This is proved by Wadsley in [10]. Throughout his paper Wadsley shows that in this situation there exist a Riemannian metric $g$ and a unit Killing vector field $\xi$ such that the trajectories of $\xi$ coincide with the leaves of $\mathcal{F}$.

Let $\theta$ be the 1-form on $M$ given by $\theta(X) = g(\xi, X)$. Then we have the following result.

PROPOSITION 4. (a) *$g$ is bundle-like with respect to $\mathcal{F}$.*

(b) *$g$ is of constant volume.*

(c) *There exists $\tau \in A^2(B)$ such that $d\theta = \pi^*\tau$. In particular, $L_\xi\theta = 0$ where $L_\xi$ is the Lie derivative with respect to $\xi$.*

PROOF. Let $(U, x, y^\alpha)$ be a flat neighborhood with $\xi = \partial/\partial x$. Let $v_\alpha = \partial/\partial y^\alpha + b_\alpha\xi$ be vector fields on $U$ orthogonal to $\xi$. Then $\{\xi, v_\alpha\}$ is the dual basis of $\{\theta, dy^\alpha\}$ and in this basis we have

$$g = \theta \otimes \theta + g_{\alpha\beta}(x, y)dy^\alpha \otimes dy^\beta.$$

An easy computation shows that

$$d\theta = -(\partial b_\alpha/\partial x)\theta \wedge dy^\alpha - (\partial b_\beta/\partial y^\alpha + b_\alpha \partial b_\beta/\partial x)dy^\alpha \wedge dy^\beta.$$

Hence $L_\xi\theta = -(\partial b_\alpha/\partial x)dy^\alpha$.

As $L_\xi g = 0$ we obtain $\partial b_\alpha/\partial x = \partial g_{\alpha\beta}/\partial x = 0$, $\alpha, \beta = 1, \cdots, q$.

Thus, $g$ is bundle-like, $L_\xi\theta = 0$ and $d\theta = -(\partial b_\beta/\partial y^\alpha)dy^\alpha \wedge dy^\beta$ where $\partial b_\beta/\partial y^\alpha$ is independent of $x$. This proves (a) and (c).

Let $L_0$ be a leaf of $\mathcal{F}$ and $z_0 \in L_0$. Let $T$ denote the $q$-ball swept out by geodesics orthogonal to $L_0$ at point $z_0$ and of length less than a small $\varepsilon > 0$. Let $\mu : \mathbf{R} \times M \to M$ be the flow of the vector field $\xi$. Since $\xi$ is Killing, $\mu_t(T)$ is also a $q$-geodesic ball of the same radious $\varepsilon$ and centered at $\mu_t(z_0)$. In particular, $\mu(\mathbf{R} \times T)$ is a saturated neighborhood of $L_0$.

Set $t_0 = \inf\{t > 0; \mu_t(z_0) = z_0\}$. Clearly $\mu_{t_0}(T) = T$ and $\mu_{t_0}$ is a generator of the cyclic group $H(L_0)$ (the holonomy group of $L_0$). Since $\xi$ is a unit vector field, the period and the length of its orbits coincide. Thus the volume of a regular leaf of $\mathcal{F}$ through $T$ is $k.\, t_0$ where $k$ is the order of $H(L_0)$. Thus, since the union of all regular leaves is a connected set, part (b) is proved.

We lose no generality in assuming that the orientation of $\xi$ is the orientation of $\mathcal{F}$ and that the volume of all regular leaves is 1. In this case $\int_{\mathcal{F}} \theta = 1$ and $\chi_{\mathcal{F}} = [\tau]$ (cf. Remark 1) and we say that $(g, \xi, \theta)$ satisfy the condition C.

3.2. We recall that on a manifold of dimension $2k+1$, a 1-form $\omega$ is said to be a contact form if $\omega \wedge (d\omega)^k \neq 0$ at each point of $M$. Then there is a unique vector field $Z$ such that $\omega(Z) = 1$ and $i_Z d\omega = 0$.

Definition 3. A circle Hausdorff foliation $\mathcal{F}$ on $M$, of even codimension, is said to be a contact foliation if there is a contact form $\omega$ on $M$ such that its associated vector field $Z$ is tangent to $\mathcal{F}$.

Concerning the existence of contact foliations we get:

Proposition 5. *Let $\mathcal{F}$ be a contact foliation. Then $\chi_{\mathcal{F}} \neq 0$.*

Proof. Let $\omega$ be a contact form with associated vector field $Z$ tangent to $\mathcal{F}$. The function $\int_{\mathcal{F}} \omega$ is constant because $i_Z d\omega = 0$ implies $d \int_{\mathcal{F}} \omega = \int_{\mathcal{F}} d\omega = 0$. As $i_Z \omega = 1$ we can assume $\int_{\mathcal{F}} \omega = 1$. Since $L_Z d\omega = 0$ we have $d\omega = \pi^* \tau$ and so $\chi_{\mathcal{F}} = [\tau]$ (cf. Remark 1). Finally $\chi_{\mathcal{F}} \neq 0$ because $\omega$ is a contact form.

We now prove the reciprocal of Proposition 3.

Theorem 2. *Let $M$ be a compact oriented manifold and $\mathcal{F}$ an oriented Hausdorff foliation by circles on $M$. The following conditions are equivalent*:

(i) $\chi_{\mathcal{F}} = 0$.

(ii) *There exists a Riemannian foliation complementary to $\mathcal{F}$.*

Proof. Let $(g, \xi, \theta)$ be a triple satisfying the condition C and let $\tau \in A^2(B)$ represent $\chi_{\mathcal{F}}$ with $d\theta = \pi^* \tau$.

Assume $\chi_{\mathcal{F}} = 0$. Then, there exists $\gamma \in A^1(B)$ such that $\tau = d\gamma$. Set $\theta' = \theta - \pi^* \gamma$. Since $\theta'(\xi) = \theta(\xi) = 1$ and $d\theta' = 0$, $\theta'$ defines a foliation $\mathcal{F}^\perp$ transverse to $\mathcal{F}$ and of codimension 1. Let $\tilde{g}$ be the Riemannian metric on $B$ induced by $g$. The Riemannian metric $g' = \theta' \otimes \theta' + \pi^* \tilde{g}$ is then bundle-like with respect to $\mathcal{F}$.

Let $(U, x, y^\alpha)$ be a flat neighborhood of $\mathcal{F}$ with $\xi = \partial / \partial x$. In these coordinates we have $\theta' = dx + c_\alpha dy^\alpha$ and $L_\xi \theta' = \xi(c_\alpha) dy^\alpha$. On the other hand, $L_\xi \theta' = 0$ and so $\xi(c_\alpha) = 0$.

Next consider a second coordinate neighborhood $(U', x', y'^\alpha)$ flat both with respect to $\mathcal{F}$ and $\mathcal{F}^\perp$. We may assume $y'^\alpha = y^\alpha$. Then $\partial/\partial x' = (\partial x/\partial x')\xi$ and $\partial/\partial y'^\alpha = \partial/\partial y^\alpha + (\partial x/\partial y'^\alpha)\xi$. As $\theta'(\partial/\partial y'^\alpha) = 0$ we have $\partial x/\partial y'^\alpha = -c_\alpha$. Hence $\partial/\partial y'^\alpha\ (\partial x/\partial x') = (\partial x/\partial x').\xi(-c_\alpha) = 0$. It follows that $g'$, whose expression in $(U', x', y'^\alpha)$ is given by

$$g' = (\partial x/\partial x')^2 dx' \otimes dx' + \pi^* \tilde{g},$$

is bundle-like with respect to $\mathcal{F}^\perp$ and so $\mathcal{F}^\perp$ is a Riemannian foliation complementary to $\mathcal{F}$.

Finally, we consider the case where $\dim M = 3$. Recall that in this case an orientable circle foliation is automatically Hausdorff [1].

THEOREM 3. *Let $M$ be a compact orientable 3-manifold and let $\mathcal{F}$ be a circle orientable foliation on $M$. The following conditions are equivalent*:

(i) *$b_1(M)$ (the first Betti number of $M$) is even.*

(ii) $\chi_{\mathcal{F}} \neq 0$.

(iii) *$\mathcal{F}$ is a contact foliation.*

(iv) *$\mathcal{F}$ does not admit any Riemannian complementary.*

PROOF. The $V$-manifold $B$ is locally of the form $D/G$ where $D$ is the unit ball in $\mathbf{R}^2$ and $G$ is a finite subgroup of SO(2). Since $D/G$ is homeomorphic to $D$, $B$ is an orientable compact 2-dimensional topological manifold (without boundary) and so $b_1(B)$ is zero or even. If $\chi_{\mathcal{F}} = 0$, $H^*(M) \cong H^*(B) \otimes H^*(S^1)$ (cf. Remark 2), and hence $b_1(M) = b_1(B) + 1$ is odd. This proves (i) $\Rightarrow$ (ii).

Consider the Gysin exact sequence

$$0 \longrightarrow H^1(B) \xrightarrow{\pi^*} H^1(M) \xrightarrow{\int_{\mathcal{F}}^*} H^0(B) \cong \mathbf{R} \xrightarrow{\mathcal{D}} H^2(B) \longrightarrow \cdots.$$

If $\chi_{\mathcal{F}} \neq 0$ then $\mathcal{D}$ is a monomorphism and $\pi^*$ an isomorphism. Hence $b_1(M)$ is zero or even. This proves (ii) $\Rightarrow$ (i).

Since (ii) $\Leftrightarrow$ (iv) and (iii) $\Rightarrow$ (ii) are true for arbitrary dimension of $M$ (cf. Proposition 5 and Theorem 2 above) it remains only to prove (ii) $\Rightarrow$ (iii).

Let $(g, \xi, \theta)$ be a tern satisfying the condition C with $d\theta = \pi^*\tau$, $\tau \in A^2(B)$, and $\chi_{\mathcal{F}} = [\tau] \neq 0$. Since $B$ is orientable there exists a $2-V$-form $\beta$ on $B$ such that $\beta \neq 0$ at each point. Hence $[\beta] \neq 0$ and so there is $r \in \mathbf{R}$, $r \neq 0$ such that $r[\beta] = [\tau]$.

Set $\omega = \theta + \pi^*\gamma$ where $d\gamma = r\beta - \tau$. Then $\omega(\xi) = \theta(\xi) = 1$ and $d\omega = r\pi^*\beta$. It follows that $\omega$ is a contact form. Moreover $i_\xi d\omega = r.\, i_\xi \pi^*\beta = 0$, so that $\xi$ is the vector field associated to $\omega$. This proves (ii) $\Rightarrow$ (iii).

REMARK 4. Since $S^3$ is simply connected, any foliation on $S^3$ is orientable. Thus, each circle foliation on $S^3$ is a contact foliation.

REMARK 5. Let $\dim M = 3$ and $b_1(M)$ odd. Then $M$ does not admit an almost regular contact form in the sense of C. B. Thomas [9], i.e., a contact form such that all trajectories of the associated vector field are closed.

## §4. An integral formula

4.1. Let $\mathcal{F}$ be a Seifert bundle and let $(g, \xi, \theta)$ be as in Proposition 4. Let $l(\xi)$ denote the volume of regular leaves of $\mathcal{F}$. Recall that $g$ induces a Riemannian metric $\bar{g}$ on the $V$-manifold $B$ such that $g = \theta \otimes \theta + \pi^* \bar{g}$. Our aim here is to study the relation between the curvatures of the Riemannian connections of $g$ and $\bar{g}$.

Let $(U, x, y^\alpha)$ be a flat neighborhood such that $U = V \times W$ where $V$ and $W$ are open sets of $\mathbf{R}$ and $\mathbf{R}^q$ respectively; $W$ is a uniformizing open set of the $V$-manifold $B$ and $\bar{g}$ is defined on $W$.

There exists a basis of 1-forms $\{\varphi^1, \cdots, \varphi^q\}$ on $W$ such that $\bar{g} = \Sigma_\alpha \varphi^\alpha \otimes \varphi^\alpha$. Let $(\omega^\alpha_\beta)$ and $(\Omega^\alpha_\beta)$ denote the corresponding matrices of connection and curvature forms. Let $\pi_U : U \to W$ be the canonical projection and set $\theta^0 = \theta$, $\theta^\alpha = \pi^*_U(\varphi^\alpha)$. Then $\{\theta, \theta^1, \cdots, \theta^q\}$ is an orthonormal basis of 1-forms on $U$ that we assume positively oriented. On the other hand, it follows from Proposition 4 that $d\theta^0$ can be written as follows:

$$d\theta^0 = \pi^*_U(A_{\alpha\beta}\varphi^\alpha \wedge \varphi^\beta), \qquad A_{\alpha\beta} = -A_{\beta\alpha}.$$

Applying now the same arguments of Kobayashi in [4] we obtain the following expressions for the connection forms $(\psi^i_j)$ and for the curvature forms $(\Psi^i_j)$ of $g$ on $U$ $(i, j = 0, 1, \cdots, q)$:

$$\begin{cases} \psi^0_0 = 0, \\ \psi^\alpha_0 = -\psi^0_\alpha = -\sum_\beta A_{\alpha\beta}\theta^\beta, \\ \psi^\alpha_\beta = \pi^*_U(\omega^\alpha) - A_{\alpha\beta}\theta^0; \end{cases}$$

$$\begin{cases} \Psi^0_0 = 0, \\ \Psi^\alpha_0 = -\Psi^0_\alpha = -\sum_{\alpha,\beta} A_{\alpha\gamma}A_{\gamma\beta}\theta^\beta \wedge \theta^0 - \sum_{\beta,\gamma} A_{\alpha\gamma;\beta}\theta^\beta \wedge \theta^\gamma, \\ \Psi^\alpha_\beta = \pi^*_U(\Omega^\alpha_\beta) - \sum_{\gamma,\delta}(A_{\alpha\beta}A_{\gamma\delta} + A_{\alpha\gamma}A_{\beta\delta})\theta^\gamma \wedge \theta^\delta - \sum_\gamma A_{\alpha\beta;\gamma}\theta^\gamma \wedge \theta^0; \end{cases}$$

where $\Sigma_\gamma A_{\alpha\beta;\gamma}\varphi^\gamma = dA_{\alpha\beta} - \Sigma_\gamma A_{\alpha\beta}\omega^\gamma_\beta + \Sigma_\gamma A_{\gamma\beta}\omega^\alpha_\gamma$.

4.2. With the notation as in the above paragraph, if $q = 2k$, the Gauss–Bonnet integrand on $W$ can be written as

$$\bar{\Omega}_w = \frac{(-1)^k}{2^{2k}k!\,\pi^k} \sum_{(\alpha_1,\cdots,\alpha_q)} \varepsilon(\alpha_1,\cdots,\alpha_q)\Omega_{\alpha_1}^{\alpha_2} \wedge \cdots \wedge \Omega_{\alpha_q-1}^{\alpha_q}.$$

REMARK 6. Since the forms $\omega_\beta^\alpha$ and $\Omega_\beta^\alpha$ are not, in general, invariant under the action of the holonomy group they are not $V$-forms on the open set of $B$ corresponding to $W$. Nevertheless, if we cover $M$ by flat neighborhoods $U = V \times W$ as above, the family $\{\bar{\Omega}_W\}$ defines a $V$-form $\bar{\Omega} \in A^q(B)$ (cf. [7]). Then $\pi_U^*(\bar{\Omega}_W) = \pi^*\bar{\Omega} \mid U$.

LEMMA 1. *Let the notation be as above. Then*

$$\frac{1}{l(\xi)} \int_M \pi^*(\bar{\Omega}) \wedge \theta = \chi_V(B)$$

*where $\chi_V(B)$ is the Euler characteristic of $B$ as $V$-manifold.*

PROOF. We have (cf. [8])

$$\int_B \bar{\Omega} = \chi_V(B).$$

Hence

$$\int_M \pi^*(\bar{\Omega}) \wedge \theta = \int_B \left( \int_{\mathcal{F}} \pi^*(\bar{\Omega}) \wedge \theta \right) = \int_B \bar{\Omega} \wedge \left( \int_{\mathcal{F}} \theta \right) = l(\xi)\chi_V(B).$$

We finally consider the case where $\dim M = 3$. Let $\eta$ be the volume element and let $R$ be the curvature tensor. Then $\eta = \theta^0 \wedge \theta^1 \wedge \theta^2$ and we have

PROPOSITION 6. *Let* $\dim M = 3$. *Then*

$$\frac{1}{2\pi l(\xi)} \int_F (K(\xi^\perp) + 3K(\xi))\eta = \chi_V(B)$$

*where $K(\xi^\perp)$ means the sectional curvature of the plane orthogonal to $\xi$ and $K(\xi)$ means the sectional curvature of any plane containing $\xi$.*

PROOF. It follows from §4.1 that the local expression of $\pi^*\bar{\Omega} \wedge \theta$ in a flat neighborhood $U = V \times W$ is

$$\pi_U^*(\bar{\Omega}_W) \wedge \theta^0 = -\frac{1}{4\pi}\pi_U^*(2\Omega_1^2) \wedge \theta^0 = -\frac{1}{2\pi}(\Psi_1^2 - 3(A_{12})^2\theta^1 \wedge \theta^2) \wedge \theta^0.$$

Hence the integral formula of Lemma 1 can be written

$$\frac{-1}{2\pi l(\xi)}\int_M (\Psi_1^2 - 3(A_{12})^2\theta^1 \wedge \theta^2) \wedge \theta^0 = \chi_V(B).$$

Let $\xi$, $X_1$, $X_2$ be the dual basis of $\theta^0$, $\theta^1$, $\theta^2$. We have

$$R(\xi, X_1, \xi, X_2) = g(R(\xi, X_1)X_1, \xi) = \Psi_1^0(\xi, X_1) = (A_{12})^2,$$

$$R(\xi, X_2, \xi, X_2) = g(R(\xi, X_2)X_2, \xi) = \Psi_2^0(\xi, X_2) = (A_{12})^2,$$

$$R(\xi, X_2, \xi, X_1) = g(R(\xi, X_2)X_1, \xi) = \Psi_1^0(\xi, X_2) = 0.$$

Thus, if $Z = \lambda X_1 + \mu X_2$ with $\lambda^2 + \mu^2 = 1$, then $R(\xi, Z, \xi, Z) = (A_{12})^2$, whence $(A_{12})^2$ is the sectional curvature $K(\xi)$ of any plane containing $\xi$.

On the other hand, $\Psi_1^2 \wedge \theta^0 = R_{112}^2\theta^0 \wedge \theta^1 \wedge \theta^2 = -R_{121}^2\theta^0 \wedge \theta^1 \wedge \theta^2$. But $R(X_1, X_2, X_1, X_2) = g(R(X_1, X_2)X_2, X_1) = \Psi_2^1(X_1, X_2) = -\Psi_1^2(X_1, X_2) = -R_{112}^2 = R_{121}^2$. It follows that $\Psi_1^2 \wedge \theta^0 = -K(\xi^\perp)\theta^0 \wedge \theta^1 \wedge \theta^2$.

This result is a generalization of theorem 2 in [6].

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